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Critical Exponents from Power Spectra

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We have simulated the two- and three-dimensional Ising models at their respective critical points with a conventional Monte Carlo algorithm. From the power spectrum of the magnetization autocorrelations we have determined the dynamic critical exponents and obtained the values $z = 2.16–2.19$ and $z = 2.05$, in agreement with the results quoted in the literature. We have also studied the power spectrum for the Kardar–Parisi–Zhang and Sun–Guo–Grant equations describing interface dynamics. Arguments similar to what was recently used to conclude that $z = 4 - \eta$ for model B in critical dynamics were applied to the Sun–Guo–Grant growth model and the known exact values for the roughening and dynamic exponents were obtained. From an analysis of the corresponding power spectrum in self-organized critical sand models one obtains a recently proposed hyperscaling relation.

KEY WORDS: Ising model; dynamic exponents; power spectra; Monte Carlo simulations; dynamics of interface growth; self-organized criticality.

1. INTRODUCTION

The static exponents for the two-dimensional Ising model are known exactly,[1] but the value of the dynamic critical exponent is still quite controversial.[2–7] Near a phase transition point the relaxation of the slowest modes diverges as $\tau \sim \xi^z$, where z is the dynamic critical exponent and ξ the correlation length (a phenomenon known as *critical slowing down*).

The value of the dynamic exponent depends on the dynamics of the system characterized by a variety of models.[2] Model A contains the Ising model with nonconserved magnetization (Glauber dynamics). Renor-

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malization group (RG) calculations have predicted that $z = 2.13$–2.18 in two dimensions and $z = 2.02$ in three dimensions.[2] Numerous Monte Carlo simulations using various techniques have been performed yielding results in agreement with the RG estimate. An Ising system with conserved magnetization (Kawasaki dynamics) belongs to model B. For this model the dynamic exponent can be obtained from a knowledge of the static critical exponents by means of the relation $z = 4 - \eta$, where η is the exponent which characterizes the decay of the correlation function at the critical point. This result was obtained from a RG calculation, and was recently also derived by Leung from an analysis of the power spectrum of the current correlations at the critical point.[8]

One way of determining z from simulations consists in fitting the magnetization autocorrelation function $C(t) = \langle m(t)\, m(0) \rangle$ to a sum of exponentials, $C(t) = \sum_i A_i e^{-t/\tau_i}$, with the largest τ_i interpreted as the critical relaxation time τ. Usually one need only retain one or two terms in the series in order to obtain good values for the dynamic exponent. Performing Monte Carlo simulations for different lattice sizes L, one obtains the functional relationship between τ and L. Using the dynamic finite-size scaling (FSS) ansats[9] $\tau \sim L^z$, one can obtain the dynamic exponent z. This method has been used to obtain the values $z = 2.17 \pm 0.04$ and $z = 2.03 \pm 0.04$ for z in two and three dimensions, respectively.[3] Fourier transforming the FSS form $C(t) = C(0) f(t/L^z)$ for the correlation function and relating the static correlations $C(0)$ to the susceptibility by means of the fluctuation-dissipation theorem, the power spectrum behavior $S(\omega) \sim \omega^{-(1+\gamma/\nu z)}$ was obtained.[10] This was tested in simulations and used to validify Suzuki's dynamic finite-size scaling ansatz.

In the neighborhood of a phase transition point the dynamic finite-size scaling ansatz $m(L, t) \sim L^{-\beta/\nu} \bar{m}(tL^{-z})$ for the magnetization is valid.[4,11] For times less than $\tau \sim L^z$ one obtains the bulklike behavior $m(L, t) \sim L^{-\beta/\nu}(tL^{-z})^{-\beta/\nu z} = t^{-\beta/\nu z}$. If the system at time $t = 0$ is prepared in a state with, for instance, $m(L, 0) = 1$, one can extract the dynamic exponent from the decay toward equilibrium, provided the critical exponents β and ν are known. Large-scale simulations using this method have recently yielded the values $z \approx 2.18$ and $z \approx 2.09$ for the dynamic exponent in two and three dimensions.[4] The concept of "damage spreading" has also been used to measure the dynamic exponent in Ising-like systems.[5,6] Measuring the time it takes for a fixed damage to propagate to the edges of the system leads to the value $z = 2.24 \pm 0.04$ in two dimensions[5] and this value is seen to differ from the previously accepted value. If one instead measures the time τ it takes for a damage to disappear, the value $z = 2.16 \pm 0.02$ is obtained[6] by (as above) using the dynamic FSS ansatz $\tau \sim L^z$.

In the present paper, motivated by the considerable interest in power spectrum studies of dynamical nonequilibrium systems, we undertake a study of the power spectrum of the magnetization autocorrelations in the Ising model. The power spectrum in the Ising model has a power law form at the phase transition point and we describe a method for determining the dynamic critical exponent from simulations of the magnetization power spectrum. Our results for the dynamic exponents for the two- and three-dimensional ferromagnetic Ising models using Glauber dynamics are $z = 2.19 \pm 0.03$ and $z = 2.05 \pm 0.05$, in agreement with the results quoted in the literature. By means of Monte Carlo simulations using Kawasaki dynamics we also analyze the two-dimensional Ising model with antiferromagnetic interactions and constant magnetization equal to zero. This system is even though it is coupled to a conserved quantity also assumed to belong to model A in critical dynamics and not to model C[12] and our result $z = 2.16 \pm 0.03$ confirms this and yields another estimate for the dynamic critical exponent.

Coarse-grained (Langevin) equations describing interface dynamics such as the Kardar–Parisi–Zhang (KPZ) equation[13] or the Sun–Guo–Grant (SGG) equation[14] can also be analyzed in terms of power spectra. For the KPZ equation we find that the power spectrum generically has a power law form characterized by an exponent related to the roughening and the dynamic exponents. It is furthermore possible to obtain the known exact values for the critical exponents characterizing the SGG equation, using arguments similar to the ones recently used by Leung in order to rederive $z = 4 - \eta$ for model B in critical dynamics.[8] A corresponding analysis of Langevin equations describing self-organized critical models[15,16] gives a relation between the power spectrum exponent and the critical exponents. In ref. 17 some of the earliest derivations of exponent identities in dynamics can be found. In the present context, combining this analysis with the nonconserving nature of the noise in sand models leads to a hyperscaling relation which was previously proposed from physical and renormalization group arguments.[18–20]

Section 2 contains a discussion of the power spectrum method. In Section 3 we present simulation results for the dynamic critical exponent. In Section 4 the ideas are applied to the KPZ and SGG equations describing interface dynamics and to self-organized critical sand models. Finally, in Section 5 we summarize and conclude.

2. THE POWER SPECTRUM METHOD

In this section we use the scaling form of the two-point correlation function at a second-order phase transition point in order to obtain an

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expression for the power spectrum of the magnetization autocorrelations. The magnetization for a d-dimensional system is

$$m(t) = V^{-1} \int_V d^d r \, \sigma(r, t) \tag{1}$$

where $\sigma(r, t)$ is the local magnetization (up or down) at position r at time t and $V = L^d$ is the volume of the system. The power spectrum of the magnetization autocorrelation function $C(t) = \langle m(t) \, m(0) \rangle$ is

$$S(\omega) = \int dt \, C(t) \, e^{i\omega t} \equiv \frac{1}{V} \tilde{G}_{mm}(k = 0, \omega) \tag{2}$$

where $\tilde{G}_{mm}(k, \omega)$ is the Fourier transform of the two-point correlation function and $\omega = 2\pi f$ with f the frequency. A simple derivation amounting to the Wiener–Khintchine relation shows that the power spectrum can be obtained from (see, e.g., ref. 21)

$$S(\omega) = \lim_{T \to \infty} \frac{1}{T} \left\langle \left| \sum_{t=1}^{T} m(t) \, e^{i\omega t} \right|^2 \right\rangle \tag{3}$$

which is the form we used to extract the power spectrum from the simulated data. As compared to determining the correlation function directly in the simulation and then performing the Fourier transform, the expression (3) is more convenient since it makes the power spectrum manifestly positive.

In the power spectrum expression (2) we now use the scaling form of the two-point correlation function at the critical point,

$$\langle \sigma(r, t) \, \sigma(0, 0) \rangle \sim r^{-(d-2+\eta)} \bar{g}(t/r^z) \tag{4}$$

Here $\bar{g}$ is a scaling function and η the critical exponent characterizing the decay of the correlations at the critical point.[22] Carrying out the integration over time yields $S(\omega) = V^{-1} \int_V d^d r \, r^{z+2-\eta-d} \bar{\phi}(\omega r^z)$, where the scaling function $\bar{\phi}$ is related to the correlation function. Finally, integrating over space leads to the power law form

$$S(\omega) \sim V^{-1} \omega^{-\mu} \tag{5}$$

where the exponent μ is given by

$$\mu = 1 + \frac{2 - \eta}{z} \tag{6}$$

The result shows that, ideally, one should obtain a characteristic $1/\omega^{\mu}$ behavior originating from the dynamic processes of the system at the critical point. The above formula (5) for the power spectrum could, alternatively, have been obtained using the Fourier-transformed form of the two-point correlation function (4) directly in Eq. (2). Using a scaling relation one obtains $\mu = 1 + \gamma/\nu z$, which corresponds to the form of the power spectrum in ref. 10. From Monte Carlo simulations on systems with known values for the critical temperature and the critical exponent η (either known exactly or determined from a finite-size scaling analysis) it is therefore possible to obtain the dynamic exponent by determining the power spectrum.

In order to take into account the finite size of the systems used in the simulations, we use the dynamic finite-size scaling expression $\tau \sim L^{z}$,[9] and make the FSS ansatz

$$S(\omega) = V^{-1}\omega^{-\mu}\psi(\omega L^{z}) \tag{7}$$

The scaling function ψ has the following limits: $\psi(x) \to \text{const}$ for $x \to \infty$, since there is no finite-size effect in the thermodynamic limit $L \to \infty$, and $\psi(x) \sim x^{\mu}$ for $x \to 0$, implying that $S(\omega)$ in the low-frequency limit approaches an L-dependent constant value. Despite of the fact that the dynamic FSS ansatz has been used to obtain the expression (7), and this expression is used in the data analysis, it will become clear from the Monte Carlo simulations (see the next section) that for system sizes as small as $L = 50$ only rather small finite-size effects are observed. For a 50×50 Ising system, using the formula (5), we obtained the value $z = 2.19 \pm 0.03$ (see Fig. 2), i.e., we obtain good values for z by using the expression (5), which was derived without using the dynamic FSS ansatz.

From the exponential decay expression $C(t) \sim e^{-t/\tau}$ for the correlation function one obtains for the power spectrum $S(\omega) \sim 2\tau/(\tau^2 + \omega^2)$, showing the existence of the characteristic time τ in the system. The reason this power law form differs from Eq. (5) is due to the fact that the exponential decay formula for the correlation function is only valid in an intermediate time interval. In the damage spreading analysis in ref. 6 this intermediate time interval was found to be difficult to determine (for an $L = 24$ system) and therefore no precise value for z could be obtained, whereas investigations of the magnetization autocorrelation function in systems of size $L = 50$–100 yield z values with uncertainties of the order of 1 or 2%.[3]

3. MONTE CARLO SIMULATIONS

In the simulations we use the standard Ising model Hamiltonian

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{8}$$

with nearest-neighbor attractive interactions and perform the simulations on a lattice with imposed periodic boundary conditions. We study the system with spin-flip Glauber dynamics, using the standard Metropolis exchange probabilities $p = \min\{1, \exp(-\beta \Delta H)\}$. This "dynamics" mimics the thermal fluctuations and the model belongs to model A in critical dynamics. The two-dimensional Ising model is convenient to investigate since for this system the critical temperature $T_c = 2.2692$ and static critical exponents are known exactly (see, e.g., ref. 1). The simulations were performed on workstations using a simple C program.

When one simulates a finite system at the critical temperature corresponding to the infinite system, the finite system is actually in the ordered phase. This is due to the fact that a pseudo critical temperature $T_c(L)$ for the finite system of size L scales as $T_c(L) \sim T_c + aL^{1/\nu}$, which is larger than the value for the infinite-system critical temperature. This is also evident from Fig. 1, which shows the time series of the magnetization

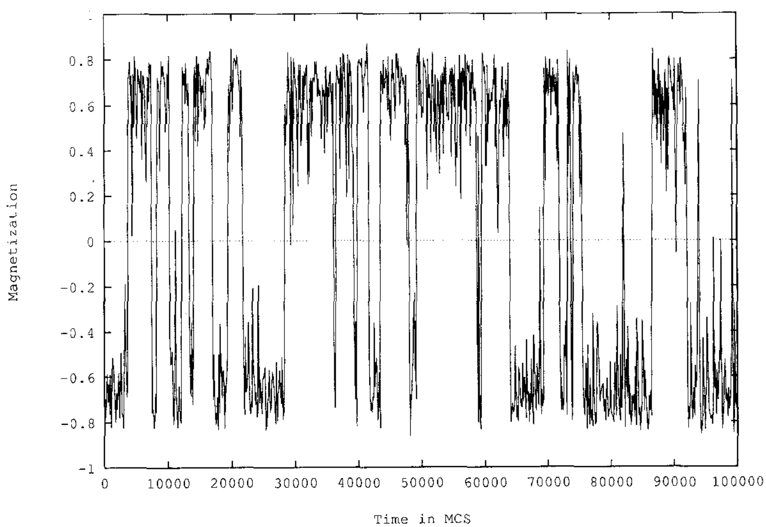

Fig. 1. The time signal of the magnetization in a 40×40 Ising model simulated at the infinite-system critical point with Glauber dynamics. One notices that most of the time the system is in one of the ordered states where the spins are fully aligned.

$m(t)=V^{-1}\sum_i \sigma_i(t)$ for a 40×40 Ising model simulated for a period consisting of 100,000 MCS (Monte Carlo steps). One notices that the system spends most of the time in a state where the spins are almost fully aligned.

Figure 2 shows the power spectrum of the correlation function for the two-dimensional Ising model with the system size $L=50$. The curve has been averaged over 100 independent runs each consisting of $2^{15}=32{,}768$ MCS. From the slope $\mu=1.80$ and using the expression (6) with $\eta=1/4$ the value $z=2.19\pm 0.03$ for the dynamic exponent is obtained. Modifications of the low-frequency part of the power spectrum, owing to the finite size of the system, are only noticeable for frequencies $f\ll\tau^{-1}\sim L^{-1}\sim 10^{-3}$–$10^{-4}$ for system sizes $L=30$–50. The power spectrum for temperatures a few percent below or above T_c shows a tendency to curve and does not fit a straight line. Reversing the procedure, i.e., determining the power spectrum for a sequence of temperatures, one can then obtain an estimate of the critical temperature. Figure 3 shows the result of a finite-size scaling analysis of the data for the two-dimensional Ising model with nearest-neighbor attractive interactions. We find the best data collapse for $z=2.19\pm 0.03$, equal to the value obtained for the $L=50$ system. The bending of the curves at low ωL^z values shows the finite-size effects.

For the Ising system with antiferromagnetic nearest-neighbor interactions, characterized by the Hamiltonian $H=\sum_{\langle ij\rangle}\sigma_i\sigma_j=-\sum_{\langle ij\rangle}(-1)^i\sigma_i(-1)^j\sigma_j$, and with the constant magnetization $m(t)=V^{-1}\sum_i\sigma_i(t)=0$, the

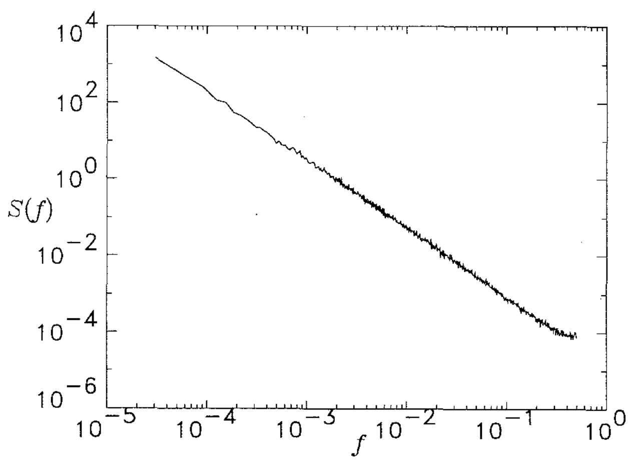

Fig. 2. The power spectrum of the correlation function for a two-dimensional Ising model. The system size is $L=50$ and the curve is an average over 100 runs. The curve fits the power law form $S(f)\sim f^{-\mu}$ with $\mu=1.80\pm 0.02$ over almost three decades, yielding $z=2.19\pm 0.03$.

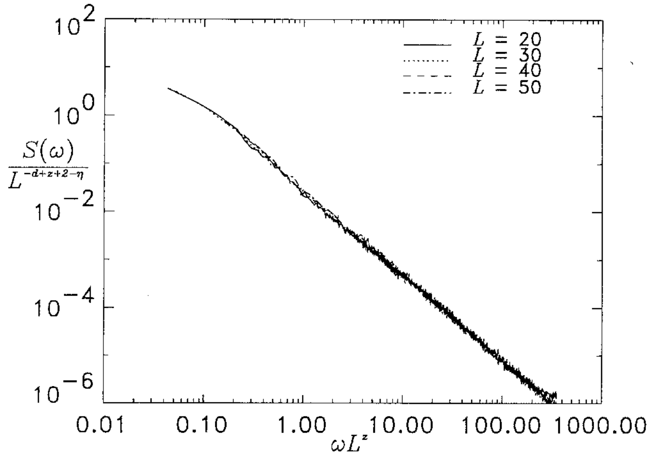

Fig. 3. Finite-size scaling analysis of the power spectra for two-dimensional Ising models simulated with Glauber dynamics. For each system size the data were averaged over 100 runs. The slope of the data collapse curve is $\mu = 1.80 \pm 0.02$ for nearly three decades, giving the estimate $z = 2.19 \pm 0.03$.

appropriate quantity to consider in the power spectrum analysis is the staggered magnetization

$$m_s(t) = V^{-1} \sum_i (-1)^i \sigma_i(t) \equiv V^{-1} \sum_i \phi_i(t) \tag{9}$$

Although the ferromagnetic (FM) and antiferromagnetic (AFM) Hamiltonians are related through $H^{\mathrm{FM}}[\sigma] = H^{\mathrm{AFM}}[\phi]$, the dynamics is different owing to the imposed constraint $\sum_i \sigma_i(t) = \sum_i (-1)^i \phi_i(t) = 0$ in the antiferromagnetic case. Since we simulate the system at the critical point where the magnetization $m(t)$ is zero, the Hamiltonian (used in the field-theoretic analysis) must possess the symmetry $m \to -m$. This means that the term relevant for model C cannot be present in the Hamiltonian and the antiferromagnetic Ising system belongs to model A in critical dynamics.[12] The staggered magnetization has the same scaling form as the magnetization in the Ising model with ferromagnetic interactions and we again obtain the expression (5) for the power spectrum.

In Fig. 4 we show the result of the FSS scaling analysis of the AFM system. Typically, we simulated the systems for $2^{15} = 32{,}768$ MCS and averaged each system size over 100 independent runs to obtain the data shown in Fig. 4. The result for the dynamic exponent is $z = 2.16 \pm 0.02$, which confirms that this model indeed does belong to model A. Furthermore, this shows that in this case the value of the dynamic exponent z does not depend on the dynamics imposed on the system. This is, for instance,

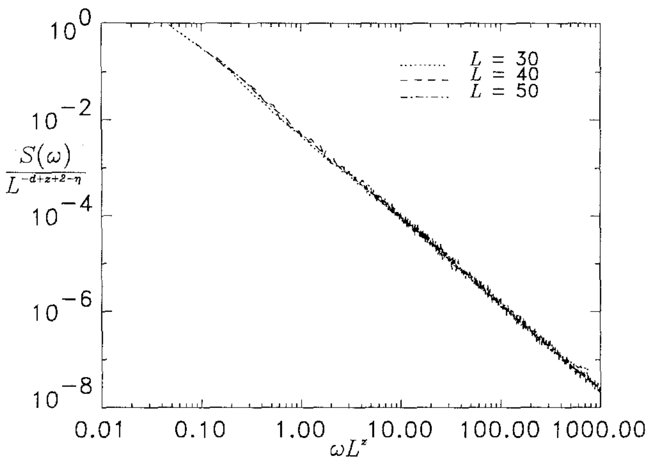

Fig. 4. Finite-size scaling analysis of the power spectra for antiferromagnetic two-dimensional Ising models simulated with conserving Kawasaki dynamics. For each system size the data were averaged over 100 runs. The slope of the data collapse curve is $\mu = 1.81 \pm 0.02$, yielding the value $z = 2.16 \pm 0.03$.

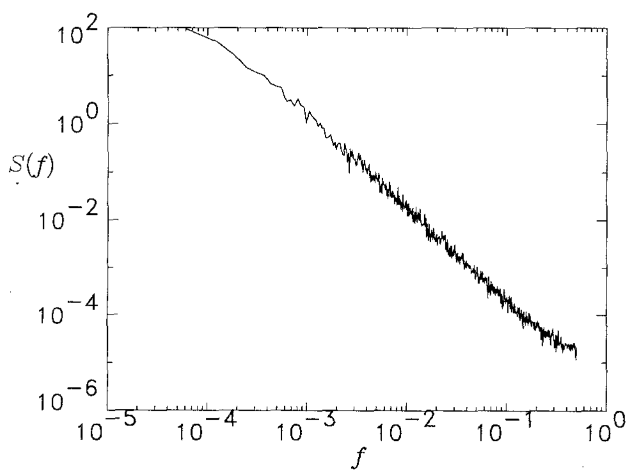

Fig. 5. The power spectrum of the correlation function for a three-dimensional Ising model of size $L = 30$ simulated with Glauber dynamics. The curve shown is an average over 20 runs each consisting of 2^{14} MCS. The slope in the middle frequency region is $\mu = 1.96 \pm 0.03$, yielding the value $z = 2.05 \pm 0.05$.

known to be the case if one uses the Swendsen–Wang–Wolf cluster updating algorithm (see, e.g., ref. 7) instead of Glauber dynamics for the Ising model. Near a second-order phase transition point the critical slowing down can be partly circumvented by using Swendsen–Wang–Wolf like cluster algorithms (see, e.g., ref. 7), owing to a very low value for the dynamic exponent z_{SW} for these algorithms.

The result for a single system size for the three-dimensional Ising model, simulated with Glauber dynamics, is shown in Fig. 5. From the slope value $\mu = 1.96$ we obtain $z = 2.05 \pm 0.05$, in agreement with the results given in the literature. The values used for the critical temperature $T_c = 4.5141$ and the static critical exponent $\eta = 0.030$ were taken from ref. 23.

4. APPLICATIONS TO "HEIGHT" MODELS

We now investigate the corresponding power spectra of the Kardar–Parisi–Zhang (KPZ) and the Sun–Guo–Grant (SGG) equations describing interface dynamics, and power spectra of self-organized critical sand models.

The KPZ equation was proposed in order to describe the long-time, long-wavelength (hydrodynamic) limit of the dynamics of nonequilibrium interface growth processes.[13] The coarse-grained KPZ equation for the height profile on a d-dimensional substrate reads

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, t) \tag{10}$$

where $\eta(x, t)$ is a Gaussian noise term with zero mean and short-range correlations

$$\langle \eta(x, t)\, \eta(x', t') \rangle = 2D\delta(x - x')\, \delta(t - t') \tag{11}$$

The ν term describes diffusion, while the nonlinear λ term determines the growth velocity at the interface. It is generally believed that the essential physics of various growth models is captured by the continuum theory proposed in ref. 13. The KPZ equation can be transformed into the Burger's equation by substituting $\phi(x, t) = -\nabla h(x, t)$. In one dimension the Burger's equation has been studied in the context of a driven diffusive system (see, e.g., the review in ref. 12; see also the discussion of the Burger's equation in ref. 17), where, using the mode-coupling expansion, the current correlations $\langle j(t)\, j(0) \rangle$ in the high-temperature homogenous phase were calculated.[24] The current correlations lead to the power spectrum behavior $S(\omega) \sim \omega^{-7/3}$ for the driven diffusive system in one dimension.

From the dynamic RG analysis of the KPZ equation[13] the height correlations were determined and generically they have the scaling form

$$\langle h(x, t)\, h(0, 0)\rangle = b^{2\chi}\langle h(x/b, t/b^z)\, h(0, 0)\rangle \sim x^{2\chi}\bar{h}(t/x^z) \tag{12}$$

where χ is the roughening exponent and z the dynamic exponent. In one dimension the exact values of the critical exponents are $\chi = 1/2$ and $z = 3/2$ and simulation results on various growth models are consistent with these values. The exponents fulfil the relation $\chi + z = 2$, which is known to be valid in any dimension and follows from a reparametrization invariance of the KPZ equation, corresponding to Galilean invariance for the Burger's equation.[17] The scaling function $\bar{h}(y)$ has the limiting values $\bar{h}(y) \to \text{const}$ for $y \to 0$, while $\bar{h}(y) \sim y^{2\chi/z}$ for $y \to \infty$. Defining the surface width as $w^2 = \langle (h - \langle h\rangle)^2\rangle$, one has the dynamic scaling form[13]

$$w(L, t) = L^\chi w(t/L^z) \sim \begin{cases} t^{\chi/z}, & t \ll L^z \\ L^\chi, & t \gg L^z \end{cases} \tag{13}$$

These scaling properties are used in simulations to estimate the critical exponents. Starting from a configuration with a flat surface, the width will initially grow as $t^{\chi/z}$. Typically, it requires large systems in order to obtain good values for the ratio χ/z. At long times the width saturates at a value proportional to L^χ, and from simulation results on different system sizes the roughening exponent can be obtained.

Corresponding to our study of the power spectrum of the magnetization autocorrelations in the Ising model, we consider the power spectrum of the height correlations

$$S(\omega) = \int_V dt \langle H(t)\, H(0)\rangle\, e^{i\omega t} \sim \langle |\tilde{H}(\omega)|^2\rangle \tag{14}$$

where the average height at time t is defined as

$$H(t) = V^{-1} \int d^d x\, h(x, t) \tag{15}$$

An analogous calculation to the one leading to Eq. (5), but now using the scaling form (12), yields

$$S(\omega) \sim V^{-1}\omega^{-\psi} \tag{16}$$

with the exponent

$$\psi = 1 + \frac{2\chi + d}{z} \tag{17}$$

Notice that no parameter has to be "fine tuned," corresponding to $T = T_c$, to obtain the long-range temporal correlations implicit in the above power law behavior for the power spectrum. Using the exact values in one dimension, one obtains $S(\omega) \sim V^{-1}\omega^{-7/3}$ and this agrees with the result obtained in ref. 24 for the current correlations in the driven diffusive system; see the discussion following Eq. (11). This connection was previously remarked in a study of fluctuations in the velocity profile $v(t) = \int d^d r \, \partial h/\partial t$ of the growing interface.[25] From the factorization of the four-point correlation function into products of two-point correlation functions the power spectrum of the velocity fluctuations was calculated with the result $\langle |\tilde{v}(\omega)|^2 \rangle \sim V^{-1}\omega^{-\alpha}$, $\alpha = (d+4)/z - 3$. From the identity $\chi + z = 2$ and the definition (15) of the average height, which yields $\tilde{v}(\omega) = -i\omega\tilde{H}(\omega)$, our result is seen to agree with the α value reported in ref. 25. Recently, using a method similar to ours, an analysis of temporal correlations of the velocity profile in growth models has been performed,[26] and in this work the predictions were confirmed in simulations on various growth models. Owing to the lower z values, compared to, for instance, the Ising model, more pronounced finite-size effects were observed for the power spectra in the growth models.

If the total height in, e.g., the KPZ equation is conserved, the appropriate quantity to consider in the power spectrum analysis is the current rather than the height. A system with a conserved height is generically described by a Langevin (continuity-like) equation

$$\frac{\partial h}{\partial t} = -\nabla \cdot j(x, t) \tag{18}$$

where the current is $j(x, t) = \nabla F[h] + \eta(x, t)$. The functional $F[h]$ depends on the system under investigation, whereas the noise term $\eta(x, t)$ has correlations given by Eq. (11). Sun, Guo, and Grant studied the case $F[h] = \nu\nabla^2 h + (\lambda/2)(\nabla h)^2$ corresponding to conserved total height for the KPZ equation. The SGG equation was investigated by RG calculations and the exponents were calculated to all orders in $\epsilon = d_c - d$ $(d_c = 2)$.[14] The equation is invariant under a reparametrization, which leads to the relation

$$\chi + z = 4 \tag{19}$$

among the exponents defined in Eq. (12). Our calculation for the KPZ equation can be repeated and the power spectrum (16) is described by the same value for the exponent, $\psi = 1 + (2\chi + d)/z$, as for the KPZ equation [see Eq. (17)].

We now follow the arguments of Leung[8] in order to determine the critical exponents characterizing surface growth according to the SGG equation. First, we consider the current along a fixed direction denoted the parallel direction. The average value of the current $\langle j \rangle$ is zero, since the system is not driven. The power spectrum of the autocorrelations of the average current $j(t) = V^{-1} \int_V d^d x \, j_{\|}(x, t)$ is

$$S_{jj}(\omega) = \int dt \langle j(t) j(0) \rangle \, e^{i\omega t} = \frac{1}{V} \tilde{G}_{jj}(k=0, \omega) \tag{20}$$

Fourier transforming the Langevin equation (18), $\partial h/\partial t = -\nabla \cdot j$, one obtains $\omega \tilde{h}_{k\omega} = k \cdot \tilde{j}_{k\omega}$, which yields the power spectrum[8]

$$S_{jj}(\omega) = \frac{1}{V} \omega^2 \lim_{k_{\|} \to 0} \frac{1}{k_{\|}^2} \tilde{G}_{hh}(k_{\|}, k_{\perp} = 0, \omega) \tag{21}$$

Here $\tilde{G}_{hh}$ is obtained by Fourier transforming equation (12), yielding $\tilde{G}_{hh}(k, \omega) = k^{-(2\chi + d + z)} \bar{G}(\omega/k^z)$, with $\bar{G}$ a scaling function. Using this expression, one obtains for the power spectrum the form

$$S_{jj}(\omega) = \frac{1}{V} \omega^2 \lim_{k \to 0} \frac{1}{k^2} k^{-(2\chi + d + z)} \bar{G}\left(\frac{\omega}{k^z}\right)$$
$$\sim \frac{1}{V} \omega^{(z - 2 - 2\chi - d)/z} \tag{22}$$

since $S_{jj}(\omega)$ is finite and positive.

A current density of the form $j(x, t) = \nabla F[h] + \eta(x, t)$ yields $\tilde{j}_{k,\omega} = -ik\tilde{F}_{k,\omega} + \tilde{\eta}_{k,\omega}$, and since $\langle \tilde{\eta}_{k\omega} \rangle = 0$, it follows that $\langle |\tilde{j}_{k=0,\omega}|^2 \rangle = \langle |\tilde{\eta}_{k=0,\omega}|^2 \rangle = \text{const}$, i.e., that $\tilde{G}_{jj}(k=0, \omega) = \langle |\tilde{j}_{k=0,\omega}|^2 \rangle = \text{const}$, and therefore from Eq. (20) that $S_{jj}(\omega) = \text{const}$.[8] Hence the SGG equation has a white noise power spectrum, but this is only compatible with Eq. (22) if the relation

$$z = 2 + 2\chi + d \tag{23}$$

among the exponents is fulfilled. Together with the identity (19), one obtains

$$z = \frac{10 + d}{3}, \qquad \chi = \frac{2 - d}{3} \tag{24}$$

in agreement with the RG calculation in ref. 14. The (height) power spectrum exponent ψ can now be calculated, $\psi = 2(7 + d)/(10 + d)$, and one obtains $\psi = 16/11$ in one dimension and $\psi = 3/2$ in two dimensions.[4]

[4] In higher dimensions (above the critical dimension $d_c = 2$) the above exponent values are not valid, and the exponents take the "classical" values $z = 4$ and $\chi = 0$.

"Self-organized critical" (SOC) sand models, i.e., extended dynamic many-body systems that self-organize into a critical state characterized by temporal and spatial long-range correlations,[15] were introduced in order to describe the ubiquitous "$1/f$ noise" observed in signals from sources ranging from the light of quasars to the flow of the river Nile. However, it was found that the spatiotemporal scaling in the SOC state does not necessarily manifest itself in nontrivial exponents for the power spectrum and, furthermore, it was found that even deterministic models can exhibit the same behavior, i.e., the criticality is not caused by, but, on the contrary, is robust with respect to noise.[16]

The basic equation describing sand models with conserving dynamics and nonconserving noise (sand added to the sandpile) is

$$\frac{\partial h}{\partial t} = -\nabla \cdot j + \eta(x, t) \tag{25}$$

where $j(x, t)$ is the current and $\eta(x, t)$ the nonconserving noise with correlations as in Eq. (11).[18] In molecular beam epitaxy (MBE) growth the surface relaxes under mass conservation and this leads to a growth equation of the type (25) with the current $j(x, t) = \nabla(\nabla^2 h)$ plus nonlinear terms.[19,20] An equation of the form (25) has been argued to exhibit generic scale invariance, i.e., correlations that under generic conditions decay algebraically in space and time.[27] Even when the dynamics is nonconserving it was found, in simulations, that the system still self-organizes into a critical state, but with the critical exponents depending on the "level" of conservation.[16]

We now determine the power spectrum for the above Langevin equation (25) (see also ref. 18 for a recent discussion of power spectra in sandpile models). Integrating the equation over space gives $\partial \tilde{h}_{k=0}/\partial t = \tilde{\eta}_{k=0}(t)$, since the current vanishes at the border, and one obtains for the power spectrum $S(\omega) \sim \langle |\tilde{H}(\omega)|^2 \rangle$ the "random walk" behavior[25,27]

$$S(\omega) \sim \frac{1}{\omega^2} \langle |\tilde{\eta}_{k=0,\omega}|^2 \rangle \propto \frac{1}{\omega^2} \tag{26}$$

yielding the value $\psi = 2$ for the exponent. The power spectrum exponent is still given by $\psi = 1 + (2\chi + d)/z$, since the scaling form for the sand model is given by Eq. (12) and therefore one obtains the hyperscaling relation

$$z = 2\chi + d \tag{27}$$

in agreement with the results in refs. 18 and 25. The relation has been derived for MBE growth models from physical[19] and renormalization group arguments.[(20),5] Using the identity (19), which holds for the nonlinear MBE equation of type (25) studied in ref. 20, one obtains the known exponents $z=(8+d)/3$ and $\chi=(4-d)/3$.

To describe a "real" sandpile, the direction in which the sand will flow has to be singled out compared to the other directions. In the RG analysis one introduces the anisotropy exponent ζ (under the rescaling $x_{\|}\to bx_{\|}$, one has $h\to b^{\chi}h$, $t\to b^{z}t$ and $x_{\perp}\to b^{\zeta}x_{\perp}$) since anisotropies are dynamically generated; see the model in ref. 18. The scaling form of the height correlations becomes

$$\langle h(x_{\|}, x_{\perp}, t)\, h(0,0)\rangle = b^{2\chi}\langle h(x_{\|}/b, x_{\perp}/b^{\zeta}, t/b^{z})\, h(0,0)\rangle \tag{28}$$

and the exponent in the power spectrum behavior $S(\omega)\sim\omega^{-\rho}$ is changed to $\rho=1+[1+2\chi+(d-1)\,\zeta]/z$. The argument yielding $\psi=2$ is still valid, i.e., one obtains $\rho=2$, leading to the relation

$$z=1+2\chi+(d-1)\,\zeta \tag{29}$$

in accordance with ref. 18. The above relation reduces to the hyperscaling relation (27) if there is no anisotropy present. In ref. 28 nonequilibrium "Flory-type" arguments were used to obtain values for the exponents for various growth and self-organized critical models, and the above relation appears as an intermediate result.

For systems with long-range noise correlations of the form

$$\langle \eta(x,t)\,\eta(x',t')\rangle = 2D\,|x-x'|^{2\rho-d}\,|t-t'|^{2\theta-1} \tag{30}$$

analyzing the behavior of the current and height power spectra as $k\to 0$, one can obtain the new scaling relations $z=2+2\chi+d-2\rho$ (for $\theta=0$), replacing Eq. (23) for the SGG equation, and $z=(2\chi+d-2\rho)/(2\theta+1)$, replacing Eq. (27) for the nonlinear MBE equation (25). For the SGG equation one investigates the current power spectrum, whereas for the nonlinear MBE equation the power spectrum of the height correlations is investigated. The current power spectrum for the SGG equation is assumed to show a white noise behavior, and combining the new scaling relation with the identity (19) yields the critical exponents $z=(10+d-2\rho)/3$ and $\chi=(2-d+2\rho)/3$, in agreement with the dynamic RG calculation in ref. 29. For the nonlinear MBE equation with exponents fulfilling Eq. (19) one obtains $z=(8+d-2\rho)/(3+2\theta)$ and $\chi=(4-d+2\rho+8\theta)/(3+2\theta)$, which agree with the results obtained in ref. 29 corresponding to the case $\theta=0$.

[5] The hyperscaling is only supposed to be valid for dimensions below the critical dimension, which will depend on the specific form of the current.

5. CONCLUSIONS

In the present paper we have determined the value of the dynamic critical exponent z from Monte Carlo simulations of the power spectrum of the magnetization autocorrelation function. The power spectrum shows a power law behavior at the phase transition point with the power spectrum exponent related to the dynamic exponent. For two-dimensional Ising models with ferro- and antiferromagnetic interactions the values $z = 2.19 \pm 0.03$ and $z = 2.16 \pm 0.03$ were obtained. For the three-dimensional Ising model we obtained $z = 2.05 \pm 0.05$. These results are in agreement with the results quoted in the literature and the uncertainties are of the same magnitude as for results obtained by other methods.

The power spectrum exponent for the Kardar–Parisi–Zhang and Sun–Guo–Grant equations describing interface dynamics was shown to be related to the dynamic and roughening exponents. Combining this relation for the SGG equation with arguments which were recently used to rederive the value of the dynamic exponent for model B in critical dynamics, we obtained the known exact values of the exponents for the SGG equation. We also investigated self-organized critical sand models by means of power spectra and obtained recently proposed hyperscaling relations. Finally, power spectra for growth models with long-range noise correlations were investigated.

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